TRANSFORMATION OF A LINEAR DIFFERENTIAL EQUATION WITH POLYNOMIAL COEFFICIENTS INTO AN INTEGRAL EQUATION WITH THE AID OF OPERATIONAL CALCULUS

(PEREKHOD OT LINEINOGO DIFFERENTSIAL'NOGO URAVNENIIA S POLINOMIAL'NYMI K KOEFFITSIENTAMI K INTEGRAL'NOMY URAVNENIIU PRI POMOSHCHI OPERATSIONNOGO ISCHISLENIIA)

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Suppose the following homogeneous linear equation is given:

$$p_0(t)\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \dots + p_n(t)y = 0 \qquad (1)$$

where, for the sake of simplicity, $p_0(t)$, $p_1(t)$, ..., $p_n(t)$ denote polynomials of order not higher than the second.

According to the [complex] differentiation theorem of the transform

$$t^m y(t) \longleftrightarrow (-1)^m p\frac{d^m}{dp^m}\left[\frac{Y(p)}{p}\right]$$

and after operational transformation, equation (1) corresponds to the following differential equation of second order:

$$pP(p)\frac{d^2}{dp^2}\left[\frac{Y(p)}{p}\right] + pQ(p)\frac{d}{dp}\left[\frac{Y(p)}{p}\right] + R(p)Y(p) = pS(p) \qquad (2)$$

where $P(p)$, $Q(p)$, $R(p)$ are polynomials of order $< n$, and $S(p)$ is a polynomial containing the initial values.

Let, for instance, $R(p)$ be a polynomial of order not less than the orders of polynomials $P(p)$ and $Q(p)$; then from (2) it follows that

$$Y(p) = \frac{pS(p)}{R(p)} - \frac{Q(p)}{R(p)}\,p\frac{d}{dp}\left[\frac{Y(p)}{p}\right] - \frac{P(p)}{R(p)}p\frac{d^2}{dp^2}\left[\frac{Y(p)}{p}\right] \qquad (3)$$

The terms of the relation (3) will be inverted separately. Assume, for the sake of simplicity, that the roots of $R(p)$ are simple. Then

$$Y(p) \to\div y(t),\ \frac{pS(p)}{R(p)} = \sum_r k_r \frac{p}{p-p_r} \div\to \sum_r k_r e^{p_r t} \tag{4}$$

$$Y(p) = p\int_0^\infty y(t)e^{-pt}dt,\ p\frac{d}{dp}\frac{Y(p)}{p} = p\int_0^\infty (-t)\,y(t)\,e^{-pt}\,dt$$

Expand $\frac{Q(p)}{R(p)}$ into simple fractions:

$$\frac{Q(p)}{R(p)} = A_0 + \sum_r \frac{A_r}{p-p_r}$$

Then

$$\frac{Q(p)}{R(p)}p\frac{d}{dp}\left[\frac{Y(p)}{p}\right] = \left(A_0 + \sum_r \frac{A_r}{p-p_r}\right)p\frac{d}{dp}\left[\frac{Y(p)}{p}\right] \equiv A_0 p\frac{d}{dp}\left[\frac{Y(p)}{p}\right] +$$

$$+\sum_r \frac{1}{p}\frac{p}{p-p_r}\left\{p\frac{d}{dp}\left[\frac{Y(p)}{p}\right]\right\}$$

But

$$A_0 p\frac{d}{dp}\left[\frac{Y(p)}{p}\right] \div\to -A_0 y(t) \tag{5}$$

and, by the convolution theorem,

$$\sum_r \frac{1}{p}\frac{p}{p-p_r}\left\{p\frac{d}{dp}\left[\frac{Y(p)}{p}\right]\right\} \div\to -\sum_r \int_0^t \tau y(\tau)e^{p_r(t-\tau)}\,d\tau \tag{6}$$

Similarly, if

$$\frac{P(p)}{R(p)} = B_0 + \sum_r \frac{B_r}{p-p_r} \tag{7}$$

then

$$\frac{P(p)}{R(p)}p\frac{d^2}{dp^2}\left[\frac{Y(p)}{p}\right] \div\to B_0 t^2 y(t) + \sum_r \int_0^t \tau^2 y(\tau)e^{p_r(t-\tau)}d\tau \tag{8}$$

Using (4), (5) - (8) yields

$$y(t)(1 - A_0 t + B_0 t^2) = \sum_r A_r \int_0^t \tau y(\tau)e^{p_r(t-\tau)}dt - \sum_r B_r \int_0^t \tau^2 y(\tau)e^{p_r(t-\tau)}dt + \sum_r K_r e^{prt}$$

This is an integral equation of the Volterra type of the second kind with a degenerate kernel, containing exponential terms.

Everything said here can be easily extended to the case where $R(p)$ has multiple roots.

Example. The equation of Laguerre

$$t\ddot{y} + (1-t)\dot{y} + ny = 0$$

is, after transformation

$$(p - p^2)\ p\frac{d}{dp}\left[\frac{Y(p)}{p}\right] - pY(p) + (n+1)Y(p) = 0$$

From this it follows that

$$p\frac{d}{dp}\left[\frac{Y(p)}{p}\right] = -\frac{p-n-1}{p^2-p}Y(p)$$

But

$$\frac{p-n-1}{p(p-1)} = \frac{n+1}{p} - \frac{n}{p-1}$$

and this means that the integral equation is

$$ty(t) = (n+1)\int_0^t y(\tau)\,d\tau - n\int_0^t y(\tau)\,e^{(t-\tau)}dt$$

or

$$ty(t) = (n+1)\int_0^t y(\tau)\,d\tau - n\,e\int_0^t y(\tau)\,e^{-\tau}dt$$

BIBLIOGRAPHY

1. Doetsch, G., *Handbuch der Laplace-Transformation*. Bd. 2, Basel und Stuttgart, 1955.

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